

Math 522 Exam 2 Solutions

1. Consider the ring $\mathbb{R}[x]$, polynomials with real coefficients. Two elements from this ring are $f(x) = x^4 - 2x^2 - 8$, $g(x) = x^3 + x^2 + 2x + 2$. Use the Euclidean algorithm to find $\gcd(f(x), g(x))$.

We use long division to find $q_1(x), r_1(x)$ satisfying $f(x) = q_1(x)g(x) + r_1(x)$, where $r_1(x)$ has degree smaller than 3 (the degree of $g(x)$). We have $q_1(x) = x - 1, r_1(x) = -3x^2 - 6$. Now we continue, finding $q_2(x), r_2(x)$ satisfying $g(x) = q_2(x)r_1(x) + r_2(x)$. It turns out that $q_2(x) = (-1/3)x - (1/3)$ and $r_2(x) = 0$, so in fact $r_1(x) = -3x^2 - 6$ is the desired gcd. Gcd's are only defined up to units, and in this ring all (nonzero) constants are units. Therefore, we can normalize the gcd to be monic. Hence, we could say that $(-1/3)r_1(x) = x^2 + 2$ is the desired gcd. This is equally correct, and looks a bit nicer.

2. Prove that for all $a, b, c \in \mathbb{N}$, $\text{lcm}(ab, ac) = a \text{lcm}(b, c)$.

Many solutions are possible. Perhaps the simplest is to use problem 4 to rewrite the LHS as $\frac{a^2bc}{\gcd(ab, ac)}$, and the RHS as $\frac{abc}{\gcd(b, c)}$. We now use the property from class, $\gcd(ab, ac) = a \gcd(b, c)$, and cancel a 's to get equality.